Lezione 17

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$ succ. esatta corta di cocatene

induce una succ. esatta lunga in coomologia

$$H^{k-1}(C) \xrightarrow{\delta} H^k(A) \xrightarrow{f_*} H^k(B) \xrightarrow{g_*} H^k(C) \xrightarrow{\delta} H^{k+1}(A)$$

Applichiamo questa costruzione alla coomologia di De Rham:

$M = U \cup V$ aperti

$$\begin{tikzcd} & U \arrow[dr, hook, "l"] & \\ V \cap U \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\ & V \arrow[ur, hook, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & \Omega^k(U) \arrow[dl, "i^*"'] & \\ \Omega^k(U \cap V) & & \Omega^k(M) \arrow[ul, "l^*"'] \arrow[dl, "m^*"] \\ & \Omega^k(V) \arrow[ul, "j^*"] & \end{tikzcd}$$

$$(\star) \quad 0 \to \Omega^k(M) \xrightarrow{(\ell^*, m^*)} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow[i_* - j_*]{} \Omega^k(U \cap V) \to 0$$

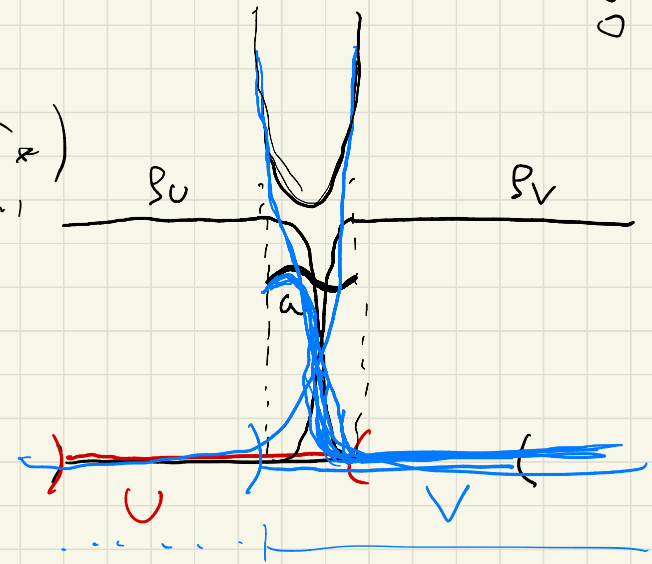
Morfismi fra cocatene

$$\begin{tikzcd}
\Omega^{k-1}(M) \arrow[r] \arrow[d, "d"] & \Omega^{k-1}(U) \oplus \Omega^{k-1}(V) \arrow[r] \arrow[d, "d"] & \Omega^{k-1}(U \cap V) \arrow[d, "d"] \\
\Omega^{k}(M) \arrow[r] \arrow[d] & \Omega^{k}(U) \oplus \Omega^{k}(V) \arrow[r] \arrow[d] & \Omega^{k}(U \cap V) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

<u>Tesi</u>: $(\star)$ ē succ. esatta corta.

$$0 \to \Omega^k(M) \xrightarrow{(\ell^*,\, m^*)} \Omega^k(U)\oplus\Omega^k(V) \xrightarrow{i_* - j_*} \Omega^k(U\cap V) \to 0$$

- (ℓ^*, m^*) iniettiva
- $\mathrm{Im}(\ell^*, m^*) \subseteq \mathrm{Ker}(i_* - j_*)$
- " " $\supseteq$ "
- $i_* - j_*$ suriettiva:



Sia $\{\rho_U, \rho_V\}$ partizione dell'unità subordinata a $\{U, V\}$

$\omega \in \Omega^k(U\cap V)$ $\quad \rho_U\omega$ estendibile ponendola nulla in V

$\mathrm{supp}(\rho_U\omega) \subseteq U \quad \Rightarrow \quad \rho_U\omega \in \Omega^k(V)$

$\rho_V\omega \in \Omega^k(U)$

$$\omega = \rho_U \omega + \rho_V \omega = \underset{\Omega^k(V)}{\underset{\cap}{\rho_U \omega}} - \underset{\Omega^k(U)}{\underset{\cap}{(-\rho_V \omega)}}$$

<u>Conseguenza</u>. Otteniamo una successione esatta lunga in coomologia

$$\dots \xrightarrow{\delta} H^k(M) \xrightarrow{(\ell^*, m^*)} H^k(U) \oplus H^k(V) \xrightarrow{i^* - j^*} H^k(U \cap V) \xrightarrow{\delta} H^{k+1}(M) \to \dots$$

SUCCESSIONE DI MAYER-VIETORIS

<u>Applicazioni</u>: Calcoliamo $H^k(S^n)$

$$S^n = \underset{U}{(S^n \setminus \{N\})} \cup \underset{V}{(S^n \setminus \{S\})}$$

$V, U \cong \mathbb{R}^n$

$U \cap V \cong \mathbb{R}^n \setminus \{pt\} \cong S^{n-1} \times \mathbb{R} \sim S^{n-1}$

<u>Prop</u>: $H^k(S^n) = \begin{cases} \mathbb{R} & \text{se } k=0 \vee k=n \\ 0 & \text{se } k \neq 0, n \end{cases}$

$n \geq 1$

dim: Induzione su n.

$n=0$ $\quad S^1$

$$0 \to \underbrace{H^0(S^1)}_{\mathbb{R}} \to \overbrace{H^0(\mathbb{R})}^{\mathbb{R}} \oplus \overbrace{H^0(\mathbb{R})}^{\mathbb{R}} \to \overbrace{H^0(S^0)}^{\mathbb{R}\times\mathbb{R}} \to$$

$$\to \underbrace{H^1(S^1)}_{\mathbb{R}} \to \underbrace{H^1(\mathbb{R})}_{=0} \oplus \underbrace{H^1(\mathbb{R})}_{=0} \to \underbrace{H^1(S^0)}_{=0} \to 0$$

$n-1 \Rightarrow n$ $\qquad k \geq 2$

$$\begin{matrix} 0 \\ \parallel \\ H^{k-1}(\mathbb{R}^n) \\ \oplus \\ H^{k-1}(\mathbb{R}^n) \\ \parallel \\ 0 \end{matrix} \to H^{k-1}(S^{n-1}) \to H^k(S^n) \to \underbrace{H^k(\mathbb{R}^n)}_{=0} \oplus \underbrace{H^k(\mathbb{R}^n)}_{=0} \to$$

$$0 \to H^{k-1}(S^{n-1}) \to H^k(S^n) \to 0$$

$$k \geq 2 \Rightarrow H^{k-1}(S^{n-1}) \cong H^k(S^n)$$

$K=1$: esercizio $\qquad H^n(S^n) = \mathbb{R}$

<u>Spazi proiettivi complessi</u> : $\mathbb{CP}^n$ $\qquad \mathbb{CP}^1 \cong S^2$

$\mathbb{CP}^n \cong S^{2n}$?

NO per $n \geq 2$

<u>Prop</u>: $H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R} & k \text{ pari } \; 0 \leq k \leq 2n \\ 0 & k \text{ dispari} \end{cases}$

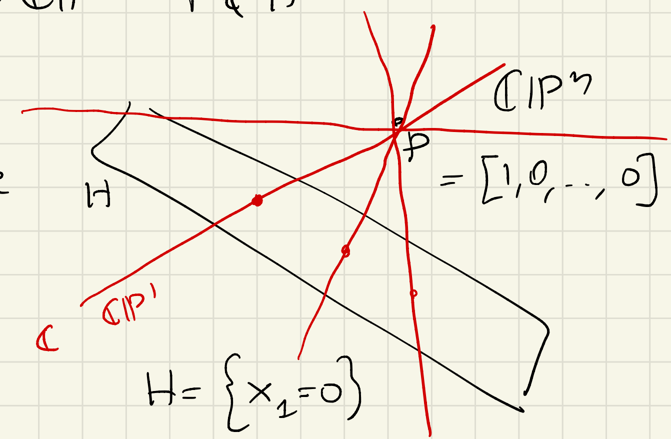
<u>dim</u>: $H \subseteq \mathbb{CP}^n$ iperpiano $\qquad p \in \mathbb{CP}^n \quad p \notin H$

$U = \mathbb{CP}^n \setminus H \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$

$\underset{\mathbb{CP}^{n-1}}{\overset{}{\underset{\cong}{H}}} \sim V = \mathbb{CP}^n \setminus \{p\}$ fibrato in rette complesse su H e si retrae per def. su H

$U \cap V =$

$\mathbb{R}^{2n} \setminus \{pt\} \cong S^{2n-1} \times \mathbb{R} \sim S^{2n-1}$



$0 \to V_1 \to \cdots \to V_n \to 0$ esatta lunga

$$\Rightarrow \sum (-1)^i \dim V_i = 0$$

$H_1(M), \ H_2(M), \ \dots, \ H_n(M)$

$$\sum (-1)^i \dim H_i(M) = \chi(M)$$

$$\chi(\mathbb{CP}^n) = n+1$$

Genericamente:

$$\underbrace{H^{k-1}(S^{2n-1})}_{\substack{= 0 \\ \text{se } k \neq 2n \\ k \neq 1}} \to H^k(\mathbb{CP}^n) \to \underbrace{H^k(\mathbb{R}^{2n})}_{\substack{= 0 \\ \text{se } k > 0}} \oplus H^k(\mathbb{CP}^{n-1}) \to \underbrace{H^{k+1}(S^{2n-1})}_{\substack{= 0 \\ \text{se } k \neq 2n-2}}$$

$$\Rightarrow H^k(\mathbb{CP}^n) \cong H^k(\mathbb{CP}^{n-1}) \qquad k \neq 2n$$

+ da fare a parte: $H^{2n}(\mathbb{C}P^n) = \mathbb{R}$ con. M.V.

<u>Cor</u>: $\mathbb{C}P^n$ non è omot. eq. S^{2n} per $n \geq 2$

$\mathbb{C}P^n$	1 0 1 0 1 0 1 0 1	numeri di Betti
S^{2n}	1 0 . . . . . . 0 1	
$S^2 \times S^2$	1 0 2 0 1	← next time
$\mathbb{C}P^2$	1 0 1 0 1	
S^4	1 0 0 0 1	1 2k 1

COOMOLOGIA A SUPPORTO COMPATTO

$$\Omega_c^k(M) = \{k\text{-forme a supp. cpt}\} \subseteq \Omega^k(M)$$

$$0 \to \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \Omega_c^2(M) \to \dots$$

$d^2 = 0$ $\qquad H_c^k(M) = \ker d^k / \operatorname{Im} d^{k-1}$

$H_c^*(M) = \oplus_c H^k(M)$ è algebra

$\omega, \eta \Rightarrow \omega \wedge \eta$

$\odot$ $M = \bigsqcup_{i \in I} M_i$ aperti $\rightarrow$ $\Omega^k(M) = \prod_{i \in I} \Omega^k(M_i)$

$\Omega_c^k(M) = \bigoplus_{i \in I} \Omega_c^k(M_i)$

Prop: M conn.

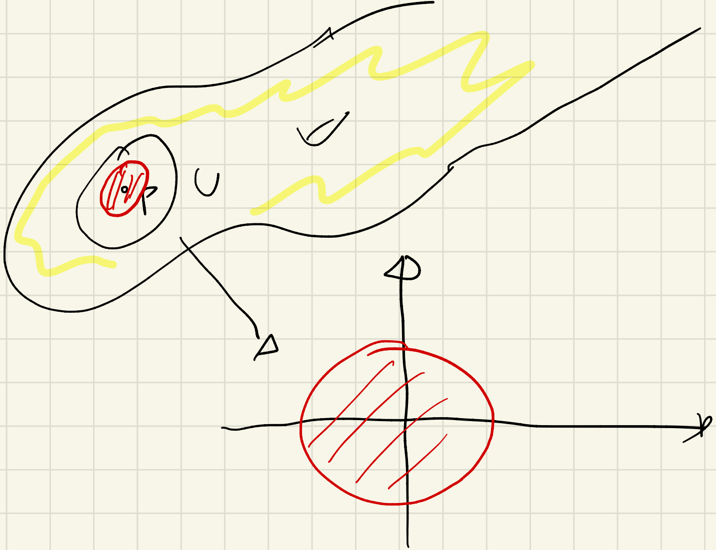
$H^0(M) = \mathbb{R}$

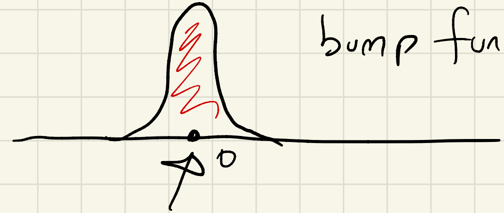
$$H_c^0(M) = \begin{cases} \mathbb{R} & \text{se } M \text{ cpt} \\ 0 & \text{se } M \text{ non cpt} \end{cases}$$

M non cpt: $H_c^0(M) = \{$funz. ~~loc.~~ cost. a supp. compatto$\} = \{0\}$

Prop: $H_c^n(M)$ non è mai banale $\quad \int_M : \Omega_c^n(M) \to \mathbb{R}$

$n = \dim M$ è suriettiva (ex)





bump functions

$$\omega = \rho(x)\, dx^1 \wedge \dots \wedge dx^n$$

$\int \omega$ qualsiasi numero possibile

$$\int_M : \Omega^n_c(M) \to \mathbb{R}$$

Se $\omega = d\eta \quad \eta \in \Omega^{n-1}_c(M)$

$$\int_M \omega = \int_{\partial M = \emptyset} \eta = 0 \quad \text{Stokes}$$

$$\int_M : H^n_c(M) \to \mathbb{R}$$

suriettiva $\Rightarrow$ $H^n_c(M)$ non banale

<u>Lemma di Poincaré</u> : $H^k_c(\mathbb{R}^n) = \begin{cases} 0 & \text{se } k<n \\ \mathbb{R} & \text{se } k=n \end{cases}$

già sappiamo che $H^0_c(\mathbb{R}^n) = 0$

$\dim H^n_c(\mathbb{R}^n) \geq 1$

Si dimostra vedendo $\mathbb{R}^n$ come $S^n - \{N\}$

$b^k_c = \dim H^k_c(M)$

$b^k_c(M)$

$$\omega \in \Omega^k_c(\mathbb{R}^n) \xrightarrow[\text{si estende}]{} \omega \in \Omega^k_c(S^n) = \Omega^k(S^n)$$

$k<n \quad \omega = d\eta$

<u>Funtorialità</u>? NO

$f: M \to N \qquad f^*: \Omega^k(N) \to \Omega^k(M)$

$\Omega^k_c(N) \not\to \Omega^k_c(M)$ (se f propria OK)

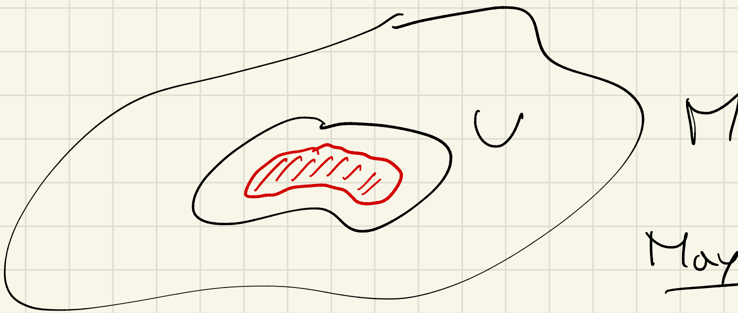
È un pochino covariante:
(a volte)

$U \subseteq M$ aperto

$i: U \hookrightarrow M$ induce $i_*: \Omega^k_c(U) \longrightarrow \Omega^k_c(M)$

$$\omega \dashrightarrow i_*(\omega)$$

estendo a zero fuori da U



<u>Mayer-Vietoris</u> (a supporto compatto)

$M = U \cup V$

$$\begin{tikzcd} & U \arrow[dr, hook, "e"] & \\ U\cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\ & V \arrow[ur, hook, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & \Omega_c(U) \arrow[dr, "e_*"] & \\ \Omega^k_c(U\cap V) \arrow[ur, "i_*"] \arrow[dr, "j_*"'] & & \Omega_c(M) \\ & \Omega_c(V) \arrow[ur, "m_*"'] & \end{tikzcd}$$

<u>Prop</u>: Otteniamo $0 \to \Omega^k_c(U\cap V) \xrightarrow{(i_*, j_*)} \Omega^k_c(U) \oplus \Omega^k_c(V) \xrightarrow{(e_* - m_*)}$

$\to \Omega^k_c(M) \to 0$

Si verifica che è esatta

Quindi si ottiene una succesione di Mayer-Vietoris

$$\to H^k_c(U \cap V) \to H^k_c(U) \oplus H^k_c(V) \to H^k_c(M) \to$$
$$H^{k+1}_c(U \cap V) \to \dots$$

Dualità di Poincaré

M varietà senza bordo orientata

$$H^k(M) \times H^{n-k}_c(M) \longrightarrow \mathbb{R}$$
$$(\omega, \eta) \longmapsto \int_M \omega \wedge \eta = \langle \omega, \eta \rangle$$

Poincaré Duality

$$PD: H^k(M) \longrightarrow H^{n-k}_c(M)^*$$
$$\omega \longmapsto (\eta \longmapsto \langle \omega, \eta \rangle)$$

$$H^k_c(M) \longrightarrow H^{n-k}(M)^*$$ ← può non essere un isomorfismo

Teo (Dualità di Poincaré) PD è isomorfismo